Generalized soldering of ± 2 helicity states in $D = 2 + 1$

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Abstract

The direct sum of a couple of Maxwell-Chern-Simons (MCS) gauge theories of opposite helicities ± 1 does not lead to a Proca theory in $D = 2 + 1$, although both theories share the same spectrum. However, it is known that by adding an interference term between both helicities we can join the complementary pieces together and obtain the physically expected result. A generalized soldering procedure can be defined to generate the missing interference term. Here we show that the same procedure can be applied to join together ± 2 helicity states in a full off-shell manner. In particular, by using second-order (in derivatives) self-dual models of helicities ± 2 (spin two analogues of MCS models) the Fierz-Pauli theory is obtained after soldering. Remarkably, if we replace the second-order models by third-order self-dual models (linearized topologically massive gravity) of opposite helicities we end up after soldering exactly with the new massive gravity theory of Bergshoeff, Hohm and Townsend in its linearized approximation.

1 Introduction

The direct sum of two chiral fermions in $D = 1 + 1$ gives rise to a full Dirac fermion, however this is not true for their bosonized versions as noticed in [1], see also [2]. Likewise, the fermionic determinant of a Dirac fermion interacting with a vector gauge field in $D = 1 + 1$ factorizes into the product of two chiral determinants but the full bosonic effective action is not the direct sum of the naive chiral effective actions as discussed in [3]. In both cases it turns out that an interference term between the opposite chirality bosonic actions is necessary to achieve the expected result, such term is provided by the so called soldering procedure.

The same procedure works in $D = 2 + 1$ if we replace chirality by helicity. In particular, the soldering of two Maxwell-Chern-Simons [4] theories of opposite helicities ± 1 leads to the Proca theory, see [5]. More generally, the ± 1 helicity modes may have different masses which leads after soldering to a Maxwell-Chern-Simons-Proca (MCSP) theory. In this case, technical problems [5] regarding a full off-shell soldering can be surmounted by defining a generalized soldering procedure [6]. In section 3 we show that such procedure can be successfully applied to fuse ± 2 helicity states of different masses $m_\pm$ with no need of using equations of motion. After soldering we obtain the Fierz-Pauli [7] theory plus a first order Chern-Simons term whose coefficient is proportional to the mass difference $m_+ - m_-$, thus generalizing a previous result [8].

A specific feature of the generalized soldering is the existence of a parameter α with a sign freedom which plays a role whenever interactions are present. In the soldering of two chiral Schwinger models that leads either to an axial ($\alpha = -1$) or to a vector ($\alpha = +1$) Schwinger model which are dual do each other. In the case of the two MCS theories, the two sign choices lead to dual interaction terms. We can have either a derivative coupling or a minimal coupling plus a Thirring term. After integration over the soldering field the dependence on the sign of α disappears which proves that they correspond to dual forms of the same interacting theory. In section 3 we couple the ± 2 helicity states with a rank two field $J_{\mu\nu}$ and show that the two signs for α lead after soldering to dual interactions similar to the spin one case. Once again integration over the soldered field lead do the same effective action $\mathcal{L}_{eff}(J_{\mu\nu})$ independent on the sign of α.

In $D = 2 + 1$ parity singlets of helicities ± 1 can be described either by the first-order self-dual model of [9] or by the second order MCS theory of [4]. Both models have their spin two counterparts, which we call $\mathcal{L}^{(1)}_{\pm 2}$ and $\mathcal{L}^{(2)}_{\pm 2}$, see [10] and [11] respectively. However, in the spin two case there is another third-order self-dual model ($\mathcal{L}^{(3)}_{\pm 2}$) with no spin one analogue. It is the quadratic truncation of the topologically massive gravity (TMG) of [4]. Although $\mathcal{L}^{(3)}_{\pm 2}$ is of third-order, it is ghost free. This is a consequence of the non-propagating nature of the Einstein-Hilbert (EH) action in $D = 2 + 1$, which allows this term to be used as a mixing term in the master approach without affecting the particle content of the interpolated theories ($\mathcal{L}^{(2)}_{\pm 2}$ and $\mathcal{L}^{(3)}_{\pm 2}$). For the same reason it is possible to jump from the second-order Fierz-Pauli theory to a fourth-order ghost free model as shown in [12], which implies the existence of a new unitary (at tree level[1]) massive gravity theory which we call henceforth BHT theory. In

[1] The residue at the massless pole generated by the Einstein-Hilbert term vanishes [13] similarly to the massless pole due to the topological Chern-Simons term in the MCS theory.

section 3 we show that the soldering of $\mathcal{L}_{+2}^{(3)}$ with $\mathcal{L}_{-2}^{(3)}$ gives rise exactly to the linearized version of the BHT theory which is, taking into account previous equivalences of soldered theories, an indication of the existence of a local dual map between the gauge invariant sectors of $\mathcal{L}_{+2}^{(3)} + \mathcal{L}_{-2}^{(3)}$ and $\mathcal{L}_{BHT}$. In the next section, as an introduction to the forthcoming sections we discuss the necessity of interference terms between opposite helicity states according to the type of self-dual model employed to describe helicity eigenstates. In section 5 we draw some conclusions.

2 Decomposition of parity doublets of spin 1 and 2 in $D = 2 + 1$

Before we start the soldering of spin two gauge theories in the next sections it is convenient to recall the description of spin 1 and spin 2 massive particles in $D = 2 + 1$ by means of non-gauge theories. In this case there will be no need of adding interference terms between opposite helicity states (parity singlets) in order to build up a parity doublet describe by just one field. We start with the spin 1 case. It is known that massive spin 1 particles are described in a covariant way by the Proca theory:

$$\mathcal{L}_P = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{m^2}{2}A^\mu A_\mu. \tag{1}$$

Throughout this work we use the signature $\eta_{\mu\nu} = (-,+,+)$. From the equations of motion of (1) one derives the transverse condition $\partial_\mu A^\mu = 0$ and the Klein-Gordon equation $(\Box - m^2)A^\mu = 0$. So we are left with 2 massive modes corresponding to the helicity states $+1$ and -1 as one can easily check by rewriting the Proca theory in a first-order form with the help of an auxiliary field B_μ,

$$\mathcal{L}_P^{(1)} = -\frac{1}{2}B^\mu B_\mu - \epsilon^{\mu\nu\alpha}B_\mu\partial_\nu A_\alpha - \frac{m^2}{2}A^\mu A_\mu. \tag{2}$$

After the redefinition, $B_\mu \to m\left(\tilde{B}_\mu + \tilde{A}_\mu\right)/\sqrt{2}$; $A_\mu \to \left(\tilde{A}_\mu - \tilde{B}_\mu\right)/\sqrt{2}$ we have two decoupled parity singlets:

$$\begin{aligned}\mathcal{L} &= -\frac{m^2}{2}\tilde{A}_\mu\tilde{A}^\mu - \frac{m}{2}\epsilon^{\mu\nu\alpha}\tilde{A}_\mu\partial_\nu\tilde{A}_\alpha - \frac{m^2}{2}\tilde{B}_\mu\tilde{B}^\mu + \frac{m}{2}\epsilon^{\mu\nu\alpha}\tilde{B}_\mu\partial_\nu\tilde{B}_\alpha \\ &= \mathcal{L}_{(+1)}^{(1)}[\tilde{A}] + \mathcal{L}_{(-1)}^{(1)}[\tilde{B}] \quad .\end{aligned} \tag{3}$$

The first-order self-dual models $\mathcal{L}_{(\pm1)}^{(1)}$ have first appeared in [9] and describe massive eigenstates of the helicity operator $\left(J\cdot P/\sqrt{P^2}\right)_{\mu\nu} = i\epsilon_{\mu\nu\gamma}\partial^\gamma/\Box$ with eigenvalues ±1. We conclude that the addition of two first order self-dual models leads to the parity invariant Proca theory in its first-order form. There is no need of adding to the non-gauge theory $\mathcal{L}_{(+1)}^{(1)} + \mathcal{L}_{(-1)}^{(1)}$ an interference term between opposite helicity states to arrive at the Proca theory which has no gauge symmetry as well. On the other hand, each first-order self-dual model is equivalent [14] to a MCS theory (second-order theory), so we expect $\mathcal{L}_P \Leftrightarrow \mathcal{L}_{MCS(+1)} + \mathcal{L}_{MCS(-1)}$ but since $\mathcal{L}_{MCS(\pm1)}$ are gauge theories it is clear that the direct sum $\mathcal{L}_{MCS(+1)} + \mathcal{L}_{MCS(-1)}$ will not lead to

the Proca theory. As explained in [5, 6], a soldering action (W_s) can be defined by the addition of an interference term quadratic in Noether currents: $W_S = W_{MCS(+1)} + W_{MCS(-1)} + W_{JJ}$ in a such way that W_S becomes exactly the Proca theory or more generally, for helicity states of different masses $m_\pm$, the Maxwell-Chern-Simons-Proca (MCSP) theory is obtained after soldering.

Now let us make a similar analysis of the spin 2 case. The spin 2 analogue of the Proca theory is the Fierz-Pauli model [7] written below in different forms for later convenience:

$$\begin{aligned}
\mathcal{L}_{FP} &= \frac{1}{2}(\sqrt{-g}R)_{hh} + \frac{m^2}{2}(h^2 - h_{\mu\nu}h^{\nu\mu}) \, , & (4)\\
&= \frac{1}{2}\Big[-\frac{\partial_\nu h_{\lambda\mu}\partial^\nu (h^{\lambda\mu} + h^{\mu\lambda})}{2} + \partial_\nu h \partial^\nu h - 2\partial_\nu h^{\lambda\nu}\partial_\lambda h \\
&+ \partial_\nu h^{\nu\lambda}\partial^\mu h_{\lambda\mu} + \frac{\partial_\nu h_{\lambda\mu}(\partial^\lambda h^{\nu\mu} + \partial^\mu h^{\lambda\nu})}{2} + m^2(h^2 - h_{\mu\nu}h^{\nu\mu}) \Big] & (5)\\
&= \frac{1}{2}T_{\mu\nu}(h)T^{\nu\mu}(h) - \frac{1}{4}T^2(h) + \frac{m^2}{2}(h^2 - h_{\mu\nu}h^{\nu\mu}) \, , & (6)
\end{aligned}$$

where $T_{\mu\nu}(h) = \epsilon_{\mu\alpha\beta}\partial^\alpha h^\beta_{\ \nu}$, $h = \eta^{\mu\nu}h_{\mu\nu}$ and $(\sqrt{-g}R)_{hh}$ stands for the Einstein-Hilbert action up to quadratic terms in the dreibein fluctuations: $e_{\mu\alpha} = \eta_{\mu\alpha} + h_{\mu\alpha}$. The field $h_{\mu\nu}$ has no symmetry in its indices. In fact, in this work all rank two fields have no specific symmetry in their indices. Symmetric and antisymmetric combinations will be denoted respectively by: $h_{(\alpha\beta)} \equiv (h_{\alpha\beta} + h_{\beta\alpha})/2$ and $h_{[\alpha\beta]} \equiv (h_{\alpha\beta} - h_{\beta\alpha})/2$. From the equations of motion of $\mathcal{L}_{FP}$ we derive the necessary constraints to describe a massive spin 2 particle, i.e., $h_{[\mu\nu]} = 0$, $\partial^\mu h_{\mu\nu} = 0 = \partial^\nu h_{\mu\nu}$, $h = 0$ and the Klein-Gordon equation $(\Box - m^2)\, h_{(\mu\nu)} = 0$. The constraints imply that we effectively have $9-7 = 2$ massive modes which will correspond to the ± 2 helicity states as follows. We rewrite the quadratic truncation of the EH term in first-order form by introducing an auxiliary tensor field $W_{\mu\nu}$ [10]:

$$\mathcal{L}^{(1)}_{FP} = \frac{m^2}{2}(W^2 - W_{\mu\nu}W^{\nu\mu}) + m\, W^{\mu\nu}\epsilon_\mu^{\ \alpha\beta}\partial_\alpha h_{\beta\nu} + \frac{m^2}{2}(h^2 - h_{\mu\nu}h^{\nu\mu}). \tag{7}$$

Redefining $W_{\mu\nu} \to (\tilde{W}_{\mu\nu} - \tilde{h}_{\mu\nu})/\sqrt{2}$ and $h_{\mu\nu} \to (\tilde{h}_{\mu\nu} + \tilde{W}_{\mu\nu})/\sqrt{2}$ one obtains two decoupled first-order self-dual models,

$$\begin{aligned}
\mathcal{L}^{(1)}_{FP} &= \frac{m^2}{2}(\tilde{W}^2 - \tilde{W}_{\mu\nu}\tilde{W}^{\nu\mu}) + \frac{m}{2}\epsilon^{\mu\alpha\beta}\tilde{W}_{\mu\nu}\partial_\alpha \tilde{W}_\beta^{\ \nu} \\
&+ \frac{m^2}{2}(\tilde{h}^2 - \tilde{h}_{\mu\nu}\tilde{h}^{\nu\mu}) - \frac{m}{2}\epsilon^{\mu\alpha\beta}\tilde{h}_{\mu\nu}\partial_\alpha \tilde{h}_\beta^{\ \nu} \\
&= \mathcal{L}^{(1)}_{+2} + \mathcal{L}^{(1)}_{-2} \quad . & (8)
\end{aligned}$$

Each of the models $\mathcal{L}^{(1)}_{\pm 2}$, first found in [10], describes an eigenstate of a spin 2 helicity operator with eigenvalues ± 2, see e.g. [10, 15]. Concluding, in both spin 1 and spin 2 cases a couple of first-order non-gauge theories of opposite helicities can be simply added up to yield a parity invariant non-gauge model containing two helicity modes. Once again, there is no need of adding any extra interference term between the opposite helicity states. However, this is not

true for the second and third-order gauge invariant actions below, which also represent $\pm$ helicity eigenstates:

$$W_{\pm 2}^{(2)} = \int d^3x \left[\frac{1}{2}T_{\mu\nu}(A)T^{\nu\mu}(A) - \frac{1}{4}T^2(A) \mp \frac{m}{2}\epsilon^{\mu\alpha\beta}A_{\mu\nu}\partial_\alpha {A_\beta}^{\nu}\right] \quad , \tag{9}$$

$$W_{\pm 2}^{(3)} = \int d^3x \left[-\frac{1}{2}T_{\mu\nu}(A)T^{\nu\mu}(A) + \frac{1}{4}T^2(A) \mp \frac{1}{2m}A_{\alpha\mu}(\Box\theta^{\alpha\gamma}E^{\beta\mu} - \Box\theta^{\alpha\mu}E^{\beta\gamma})A_{\gamma\beta}\right], \tag{10}$$

where $T_{\mu\nu}(A) = {\epsilon_\mu}^{\gamma\delta}\partial_\gamma A_{\delta\nu}$ and

$$E_{\mu\nu} = \epsilon_{\mu\nu\gamma}\partial^\gamma \quad ; \quad \Box\theta^{\mu\nu} = \eta^{\mu\nu}\Box - \partial^\mu\partial^\nu . \tag{11}$$

The second-order model $W_{\pm 2}^{(2)}$, which appeared before in [10, 11] is the spin two analogue of the MCS theory. It is invariant under the local transformations $\delta A_{\mu\nu} = \partial_\mu \xi_\nu$. The quadratic truncation of the topologically massive gravity (TMG) of [4], $W_{\pm 2}^{(3)}$, is invariant under the more general local transformations $\delta A_{\mu\nu} = \partial_\mu \xi_\nu + \epsilon_{\mu\nu\gamma}\Lambda^\gamma$. The Einstein-Hilbert term appears with the correct sign in $W_{\pm 2}^{(2)}$ in contrast to $W_{\pm 2}^{(3)}$. Both models are unitary and can be deduced from $W_{\pm 2}^{(1)} = \int d^3x \, \mathcal{L}_{\pm 2}^{(1)}$ via master action [16]. There is a local dual map connecting correlation functions in $W_{\pm 2}^{(1)}$ with correlation functions of gauge invariant objects in $W_{\pm 2}^{(2)}$ and $W_{\pm 2}^{(3)}$ up to contact terms [16]. In the next section we solder $W_{+2}^{(2)}$ and $W_{-2}^{(2)}$, the case of $W_{\pm 2}^{(3)}$ will be treated in section 4.

3 Soldering of $W_{+2}^{(2)}$ and $W_{-2}^{(2)}$

We start with the second-order opposite helicity models:

$$W_{+2}^{(2)}[A] = \int d^3x \left[\frac{1}{2}T_{\mu\nu}(A)T^{\nu\mu}(A) - \frac{1}{4}T^2(A) + \frac{m_+}{2}\epsilon^{\mu\gamma\rho}A_{\mu\nu}\partial_\gamma {A_\rho}^{\nu} + \gamma_+ \, \epsilon^{\mu\gamma\rho}J_{\mu\nu}\partial_\gamma {A_\rho}^{\nu}\right], \tag{12}$$

$$W_{-2}^{(2)}[B] = \int d^3x \left[\frac{1}{2}T_{\mu\nu}(B)T^{\nu\mu}(B) - \frac{1}{4}T^2(B) - \frac{m_-}{2}\epsilon^{\mu\gamma\rho}B_{\mu\nu}\partial_\gamma {B_\rho}^{\nu} + \gamma_- \epsilon^{\mu\gamma\rho}J_{\mu\nu}\partial_\gamma {B_\rho}^{\nu}\right]. \tag{13}$$

The masses $m_\pm$ can take arbitrary positive values. As in the spin 1 case we have added linear couplings with a rank two tensor $J_{\mu\nu}$. The interaction terms are such that the global shifts $\delta A_{\mu\nu} = \omega_{\mu\nu}$; $\delta B_{\mu\nu} = \tilde{\omega}_{\mu\nu}$ and the local transformations $\delta A_{\mu\nu} = \partial_\mu \xi_\nu$; $\delta B_{\mu\nu} = \partial_\mu \tilde{\xi}_\nu$ which are symmetries of the first two terms of (12) and (13), are preserved. Furthermore, those are the natural interaction terms when $W_{\pm 2}^{(2)}$ are deduced from $W_{\pm 2}^{(1)}$ via master action [16]. The coupling constants $\gamma_\pm$ are in principle arbitrary but special cases will be treated latter on. Both $W_{\pm 2}^{(2)}$ are also invariant under $\delta_\phi J_{\mu\nu} = \partial_\mu \phi_\nu$.

The basic idea of the soldering procedure is to lift the global shift symmetry to a local symmetry and tie the fields $A_{\mu\nu}$ and $B_{\mu\nu}$ together by imposing that their transformations are proportional do each other:

$$\delta A_{\mu\nu} = \omega_{\mu\nu} \quad ; \quad \delta B_{\mu\nu} = \alpha \, \omega_{\mu\nu} \quad , \tag{14}$$

where α is so far an arbitrary constant. From (12) and (13) we derive

$$\delta\left(W^{(2)}_{+2}[A]+W^{(2)}_{-2}[B]\right)=\int d^3x\,J^{\mu\nu\lambda}\partial_\nu\omega_{\lambda\mu} \tag{15}$$

with

$$J^{\mu\nu\lambda}=C^{\mu\nu\lambda}_{\ \ \ \ \rho\beta\gamma}\partial^\beta g^{\gamma\rho}+\epsilon_\gamma^{\ \nu\lambda}f^{\gamma\mu} \tag{16}$$

and

$$C^{\mu\nu\lambda}_{\ \ \ \ \rho\beta\gamma} = -\frac{1}{2}\epsilon^{\mu\nu\lambda}\epsilon_{\rho\beta\gamma}+\epsilon_\rho^{\ \nu\lambda}\epsilon^\mu_{\ \beta\gamma} \tag{17}$$

$$g_{\mu\nu} = A_{\mu\nu}+\alpha B_{\mu\nu} \tag{18}$$

$$f_{\mu\nu} = m_+A_{\mu\nu}-\alpha m_-B_{\mu\nu}+(\gamma_++\alpha\gamma_-)J_{\mu\nu} \tag{19}$$

In a first step Noether procedure we cancel the variation (15) introducing auxiliary fields $H_{\mu\nu\lambda}$ such that

$$\delta H_{\mu\nu\lambda}=-\partial_\nu\omega_{\lambda\mu}\quad . \tag{20}$$

Therefore

$$\delta\left(W^{(2)}_{+2}[A]+W^{(2)}_{-2}[B]+\int d^3x\,J^{\mu\nu\lambda}H_{\mu\nu\lambda}\right)=\int d^3x\,\delta J^{\mu\nu\lambda}H_{\mu\nu\lambda} \tag{21}$$

Since $\delta J^{\mu\nu\lambda}=(1+\alpha^2)C^{\mu\nu\lambda}_{\ \ \ \ \rho\beta\gamma}\partial^\beta\omega^{\gamma\rho}+\epsilon^{\gamma\nu\lambda}(m_+-\alpha^2m_-)\omega_\gamma^{\ \mu}$, if we choose

$$\alpha=\pm\sqrt{\frac{m_+}{m_-}}\quad , \tag{22}$$

we have

$$\delta J^{\mu\nu\lambda} = (1+\alpha^2)C^{\mu\nu\lambda}_{\ \ \ \ \rho\beta\gamma}\partial^\beta\omega^{\gamma\rho} \tag{23}$$

$$= -(1+\alpha^2)C^{\mu\nu\lambda}_{\ \ \ \ \rho\beta\gamma}\delta H^{\rho\beta\gamma}. \tag{24}$$

From (21) and (24) we deduce $\delta W^{(2)}_S=0$ where the soldered action is given by:

$$W^{(2)}_S=W^{(2)}_{+2}[A]+W^{(2)}_{-2}[B]+\int d^3x\,\left[J^{\mu\nu\lambda}H_{\mu\nu\lambda}+\frac{(1+\alpha^2)}{2}C^{\mu\nu\lambda}_{\ \ \ \ \rho\beta\gamma}H^{\rho\beta\gamma}H_{\mu\nu\lambda}\right] \tag{25}$$

After the elimination of the auxiliary fields through their algebraic equations of motion we end up with

$$W^{(2)}_S=W^{(2)}_{+2}[A]+W^{(2)}_{-2}[B]-\int d^3x\,\frac{\left[J^*_{\mu\nu}J^{*\nu\mu}-(J^*)^2\right]}{8(1+\alpha^2)}\quad , \tag{26}$$

where $J^*=\eta^{\mu\nu}J^*_{\mu\nu}$ with

$$J^*_{\mu\nu}=\epsilon_\mu^{\ \gamma\lambda}J_{\nu\gamma\lambda}=2\,T_{\mu\nu}(g)-\eta_{\mu\nu}T(g)-2f_{\mu\nu} \tag{27}$$

The reader can check that (26) is invariant under (14) by using (15) and (23) where α is given in (22). After some algebra we can rewrite $W_S^{(2)}$ in a more explicit form:

$$
\begin{aligned}
W_S^{(2)} &= \frac{1}{2(1+\alpha^2)}\int d^3x \, \big[\sqrt{-g}R|_{hh} + (m_+ - m_-)\epsilon^{\mu\gamma\rho}h_{\mu\nu}\partial_\gamma h_\rho{}^{\nu} \\
&+ m_+ m_- \left(\tilde{h}^2 - \tilde{h}_{\mu\nu}\tilde{h}^{\nu\mu}\right) + (\alpha\gamma_+ - \gamma_-)\,\epsilon^{\mu\gamma\rho}J_{\mu\nu}\partial_\gamma h_\rho{}^{\nu}\Big] . \qquad (28)
\end{aligned}
$$

We have introduced the combinations

$$
h_{\mu\nu} = \alpha A_{\mu\nu} - B_{\mu\nu} \quad , \qquad (29)
$$

$$
\tilde{h}_{\mu\nu} = h_{\mu\nu} + \frac{(\gamma_+ + \alpha\gamma_-)}{\alpha m_-} J_{\mu\nu} \quad . \qquad (30)
$$

The invariance under (14) has forced the action $W_S^{(2)}$ to depend only upon the combination $h_{\mu\nu} = \alpha A_{\mu\nu} - B_{\mu\nu}$, invariant under (14), which is called the soldering field. In particular, if $m_+ = m_-$ the soldered action corresponds exactly to the Fierz-Pauli theory [7] which is known to describe massive spin 2 particles in arbitrary D-dimensional spaces. It is remarkable that the nontrivial Fierz-Pauli mass term has been generated out of mass terms of Chern-Simons type appearing in $W_{\pm 2}^{(2)}$. If we drop the interactions ($J_{\mu\nu} = 0$) and set $h = 0 = h_{[\mu\nu]}$, which certainly hold on-shell, at action level we recover the soldered action of [8] obtained for $m_+ = m_-$. The mass split $m_+ - m_- \neq 0$ is responsible for the parity breaking Chern-Simons term in (28) analogously to the spin 1 case [6].

Regarding the interactions, besides the derivative coupling (last term in (28)), already present in $W_{\pm 2}^{(2)}$ before soldering, there appears now a linear coupling through the combination $\tilde{h}_{\mu\nu}$ such that the symmetry $\delta_\phi J_{\mu\nu} = \partial_\mu \phi_\nu$ of $W_{\pm 2}^{(2)}$ is maintained if we transform the soldering field accordingly. Namely, $\delta_\phi W_S^{(2)} = 0$ under:

$$
\delta_\phi J_{\mu\nu} = \partial_\mu \phi_\nu \quad ; \quad \delta_\phi h_{\mu\nu} = -\frac{(\gamma_+ + \alpha\gamma_-)}{\alpha\, m_-}\partial_\mu\phi_\nu . \qquad (31)
$$

The soldered action $W_S^{(2)}$ depends explicitly through its interaction terms on the sign choice of α defined in (22). In order to check if we do really have any physical consequence of the sign freedom we proceed as in [6] and integrate over the soldering field $h_{\mu\nu}$ in the path integral and derive an effective action $\mathcal{L}_{eff}[J_{\mu\nu}]$. Although the integral is Gaussian, the fact that $h_{\mu\nu}$ has no symmetry in its indices makes its propagator quite complicate. Our final result contains even and odd parity terms:

$$
\begin{aligned}
&(\quad -2 \quad)\mathcal{L}_{eff}[J_{\mu\nu}] \\
&= J^{\mu\nu}\left[\Box\left(P^{(2)}_{even}\right)_{\mu\nu}^{\ \ \delta\epsilon}\left(\frac{\gamma_+^2}{\Box - m_+^2} + \frac{\gamma_-^2}{\Box - m_-^2}\right) + \sqrt{\Box}\left(P^{(2)}_{odd}\right)_{\mu\nu}^{\ \ \delta\epsilon}\left(\frac{m_+\gamma_+^2}{\Box - m_+^2} - \frac{m_-\gamma_-^2}{\Box - m_-^2}\right)\right. \\
&+ \left.\left(\frac{\gamma_+^2}{m_+^2} + \frac{\gamma_-^2}{m_-^2}\right)\frac{\Box\theta_{\mu\nu}\theta^{\delta\epsilon}}{2} + \left(\frac{\gamma_+^2}{m_+} - \frac{\gamma_-^2}{m_-}\right)\left(\theta_{\mu\nu}E^{\epsilon\delta} - \frac{\partial_\nu\partial^\epsilon}{\Box}E_\mu{}^{\delta}\right)\right] J_{\delta\epsilon} \\
&. \qquad (32)
\end{aligned}
$$

The spin two projection operators are given by:

$$\left(P_{even}^{(2)}\right)_{\mu\nu}^{\quad\delta\epsilon} = \frac{1}{2}\left(\theta_\mu^{\ \delta}\theta_\nu^{\ \epsilon} + \theta_\mu^{\ \epsilon}\theta_\nu^{\ \delta} - \theta_{\mu\nu}\theta^{\delta\epsilon}\right) \quad , \tag{33}$$

$$\left(P_{odd}^{(2)}\right)_{\mu\nu}^{\quad\delta\epsilon} = \frac{1}{4\sqrt{\Box}}\left(E_\mu^{\ \delta}\theta_\nu^{\ \epsilon} + E_\mu^{\ \epsilon}\theta_\nu^{\ \delta} + E_\nu^{\ \epsilon}\theta_\mu^{\ \delta} + E_\nu^{\ \delta}\theta_\mu^{\ \epsilon}\right) \quad , \tag{34}$$

A detailed comparison with the spin 1 case, see the second reference of [6], reveals that the first two terms of (32) are remarkably similar to their spin 1 counterparts which have a Maxwell-Chern-Simons structure. In the same fashion as the differential operator in the Chern-Simons term is the square root of the differential operator in the Maxwell term ($E_{\mu\nu}E^{\nu\gamma} = \Box\theta_\mu^{\ \gamma}$) we have $\left(P_{odd}^{(2)}\right)_{\mu\nu}^{\quad\delta\epsilon}\left(P_{odd}^{(2)}\right)_{\delta\epsilon}^{\quad\gamma\rho} = \left(P_{even}^{(2)}\right)_{\mu\nu}^{\quad\gamma\rho}$.

As expected, the effective action is invariant under the original symmetry $\delta_\phi J_{\mu\nu} = \partial_\mu\phi_\nu$ of $W_{\pm 2}^{(2)}$ since $E_\mu^\nu\partial_\nu\phi = \theta_\mu^\nu\partial_\nu\phi$. Moreover, in the special case where the couplings satisfy

$$\gamma_+^2 = \frac{m_+}{m_-}\gamma_-^2 = \alpha^2\gamma_-^2 \quad , \tag{35}$$

the effective theory only depends upon $J_{(\mu\nu)}$ and consequently it is invariant under any anti-symmetric local shift $\delta_\Lambda J_{\mu\nu} = \epsilon_{\mu\nu\gamma}\Lambda^\gamma$. Indeed, we have checked that if $\gamma_+ = \pm\alpha\gamma_-$ it follows that $\delta_\Lambda W_S^{(2)} = 0$ under, respectively,

$$\delta_\Lambda J_{\mu\nu} = \epsilon_{\mu\nu\gamma}\Lambda^\gamma \tag{36}$$

$$\delta_\Lambda h_{\mu\nu} = -\frac{\gamma_-}{m_-}\left[(1\pm 1)\epsilon_{\mu\nu\gamma}\Lambda^\gamma + \frac{m_+ \mp m_-}{m_+m_-}\partial_\mu\Lambda_\nu\right]. \tag{37}$$

We also have the discrete symmetry $(m_+, m_-, \gamma_+, \gamma_-) \to (-m_-, -m_+, \gamma_-, \gamma_+)$ in $\mathcal{L}_{eff}[J]$ which amounts, before soldering, to interchange $W_{\pm 2}^{(2)} \rightleftharpoons W_{\mp 2}^{(2)}$.

As in the previous soldering cases [6], the dependence on the sign of α disappears completely after integration over the soldering field $h_{\mu\nu}$. In particular[2], if $\gamma_- = -\gamma_+ \equiv \gamma$ and $m_+ = m_-$, the two choices $\alpha = \pm 1$ lead to $\mathcal{L}_S^{\alpha=+1}(j) = \mathcal{L}_S(0) - 2\gamma J^{\mu\nu}\epsilon_{\nu\gamma\rho}\partial^\nu h^{\gamma\rho}$ and $\mathcal{L}_S^{\alpha=-1}(j) = \mathcal{L}_S(0) + 4m\gamma\left(J\,h - J_{\mu\nu}h^{\nu\mu}\right) + 4\gamma^2\left(J^2 - J_{\mu\nu}J^{\nu\mu}\right)$. Thus, the sign freedom of α gives rise to dual theories as in the spin 1 case in $D = 2+1$ and in the soldering of two Chiral Schwinger models in $D = 1+1$.

Finally, we mention that in the second part of [16] the equivalence of (28) and the gauge invariant sector of $W_{+2}^{(2)} + W_{-2}^{(2)}$ has been proved at quantum level, see also [17, 18]. So, the soldering procedure has led once more to a physically equivalent (dual) theory.

4 Soldering of $W_{+2}^{(3)}$ and $W_{-2}^{(3)}$

For sake of simplicity we drop interactions in this section and begin with the following third-order self-dual models of helicities ± 2 which correspond to quadratic truncations of topologically massive gravity, see (10),

[2] The case where opposite helicity states have opposite derivative couplings ($\gamma_+ = -\gamma_-$) naturally appears when we obtain $W_{\pm 2}^{(2)}$ from $W_{\pm 2}^{(1)}$ via master action [16]

$$W^{(3)}_{+2}[A] = \int d^3x \left[-\frac{1}{2} T_{\mu\nu}(A) T^{\nu\mu}(A) + \frac{1}{4} T^2(A) - \frac{1}{2m_+} A_{\alpha\mu} (\Box\theta^{\alpha\gamma} E^{\beta\mu} - \Box\theta^{\alpha\mu} E^{\beta\gamma}) A_{\gamma\beta} \right] \quad (38)$$

$$W^{(3)}_{-2}[B] = \int d^3x \left[-\frac{1}{2} T_{\mu\nu}(B) T^{\nu\mu}(B) + \frac{1}{4} T^2(B) + \frac{1}{2m_-} B_{\alpha\mu} (\Box\theta^{\alpha\gamma} E^{\beta\mu} - \Box\theta^{\alpha\mu} E^{\beta\gamma}) B_{\gamma\beta} \right]. \quad (39)$$

Now, we follow basically the same steps which have led us from (12) and (13) to (26). We require the soldered theory to be invariant under $\delta A_{\mu\nu} = \omega_{\mu\nu}$; $\delta B_{\mu\nu} = \tilde{\alpha}\omega_{\mu\nu}$ with $\tilde{\alpha}$ a constant to be determined. So we derive

$$\delta \left(W^{(3)}_{+2}[A] + W^{(3)}_{-2}[B] \right) = \int d^3x \, J^{\mu\nu\lambda} \partial_\nu \omega_{\lambda\mu} \quad (40)$$

where now, compare with (16), the Noether current contains first and second derivatives terms, i.e.,

$$J^{\mu\nu\lambda} = -C^{\mu\nu\lambda}_{\rho\beta\gamma} \partial^\beta \tilde{g}^{\gamma\rho} - \frac{1}{2} D^{\nu\lambda\mu\gamma\rho} \tilde{f}_{\gamma\rho}, \quad (41)$$

where $C^{\mu\nu\lambda}_{\rho\beta\gamma}$ is defined as in (17) while

$$\begin{aligned} \tilde{g}_{\mu\nu} &= A_{\mu\nu} + \tilde{\alpha} B_{\mu\nu} && (42) \\ \tilde{f}_{\mu\nu} &= \frac{A_{\mu\nu}}{m_+} - \tilde{\alpha} \frac{B_{\mu\nu}}{m_-} && (43) \\ D^{\nu\lambda\mu\gamma\rho} &= \epsilon^{\beta\lambda\nu} \left(2 E_\beta{}^\gamma E^{\rho\mu} - E_\beta{}^\mu E^{\rho\gamma} - \Box\theta^{\gamma\rho} \eta^\mu{}_\beta \right). && (44) \end{aligned}$$

Introducing auxiliary fields which transform as $\delta H_{\mu\nu\lambda} = -\partial_\mu \omega_{\nu\lambda}$ we deduce

$$\delta \left(W^{(3)}_{+2}[A] + W^{(3)}_{-2}[B] + \int d^3x \, J^{\mu\nu\lambda} H_{\mu\nu\lambda} \right) = \int d^3x \, \delta J^{\mu\nu\lambda} H_{\mu\nu\lambda}. \quad (45)$$

However, we have now

$$\delta J^{\mu\nu\lambda} = (1 + \tilde{\alpha}^2) C^{\mu\nu\lambda}_{\rho\beta\gamma} \partial^\beta \omega^{\gamma\rho} - \frac{1}{2} D^{\nu\lambda\mu\gamma\rho} \left(\frac{1}{m_+} - \frac{\tilde{\alpha}^2}{m_-} \right) \omega_{\gamma\rho}. \quad (46)$$

As in the last section, we suppress the last term above by fixing $\tilde{\alpha}$ up to a sign

$$\tilde{\alpha} = \pm \sqrt{\frac{m_-}{m_+}} \quad . \quad (47)$$

Consequently

$$\begin{aligned} \delta J^{\mu\nu\lambda} &= -(1 + \tilde{\alpha}^2) C^{\mu\nu\lambda}_{\rho\beta\gamma} \partial^\beta \omega^{\gamma\rho} && (48) \\ &= (1 + \tilde{\alpha}^2) C^{\mu\nu\lambda}_{\rho\beta\gamma} \delta H^{\rho\beta\gamma}. && (49) \end{aligned}$$

Note the sign difference to (23) and (24). This is due to the "wrong" sign of the Einstein-Hilbert term in (38) and (39). Thus, after elimination of the auxiliary fields we have, compare with (26),

$$W_S^{(4)} = W_{+2}^{(3)}[A] + W_{-2}^{(3)}[B] + \int d^3x \, \frac{\left[J^*_{\mu\nu} J^{*\nu\mu} - (J^*)^2\right]}{8(1+\tilde{\alpha}^2)} \quad , \tag{50}$$

where now

$$J^*_{\mu\nu} = \epsilon_\mu{}^{\gamma\lambda} J_{\nu\gamma\lambda} = 2\, T_{\mu\nu}(\tilde{g}) - \eta_{\mu\nu} T(\tilde{g}) - V_{\mu\nu} \tag{51}$$

with

$$\begin{aligned} V^{\mu\nu} &= -\frac{1}{2}\epsilon^\mu{}_{\gamma\rho}\, D^{\gamma\rho\nu\epsilon\delta}\, \tilde{f}_{\epsilon\delta} && (52) \\ &= \left(2E^{\mu\epsilon}E^{\delta\nu} + E^{\mu\nu}E^{\epsilon\delta} - \Box\eta^{\mu\nu}\theta^{\epsilon\delta}\right) \tilde{f}_{\epsilon\delta}. && (53) \end{aligned}$$

Rewriting the fields in term of the soldering invariant combination $h_{\mu\nu} = \tilde{\alpha} A_{\mu\nu} - B_{\mu\nu}$:

$$\begin{aligned} \tilde{f}_{\mu\nu} &= \frac{A_{\mu\nu}}{m_+} - \tilde{\alpha}\frac{B_{\mu\nu}}{m_-} = \frac{\tilde{\alpha}}{m_-} h_{\mu\nu} && (54) \\ g_{\mu\nu} &= A_{\mu\nu} + \tilde{\alpha} B_{\mu\nu} = \tilde{\alpha}\, A_{\mu\nu} - h_{\mu\nu}. && (55) \end{aligned}$$

It turns out that $W_S^{(4)}$ only depends on $h_{\mu\nu}$. In particular, all the fourth-order terms in $W_S^{(4)}$ stem from the combination:

$$\int \frac{d^3x}{8(1+\tilde{\alpha}^2)} \left(V_{\mu\nu}V^{\nu\mu} - V^2\right) = \int \frac{d^3x}{4(1+\tilde{\alpha}^2)} h_{(\mu\nu)} \left(\frac{2\,\theta^{\mu\epsilon}\theta^{\nu\delta} - \theta^{\mu\nu}\theta^{\epsilon\delta}}{m_+ m_-}\right) \Box^2 h_{(\epsilon\delta)}. \tag{56}$$

In deriving (56) from (53) we have used integration by parts, the identities $E_{\mu\gamma}E^\gamma{}_\nu = \Box\theta_{\mu\nu}$; $E_{\mu\nu}\theta^{\nu\gamma} = E_\mu{}^\gamma$; $\theta_{\mu\nu}\theta^{\nu\gamma} = \theta_\mu{}^\gamma$, equation (54) and $(\tilde{\alpha}/m_-)^2 = 1/m_+m_-$. After collecting all terms in (50) we can write the corresponding soldered Lagrangian density $\mathcal{L}_S^{(4)}$ as:

$$\mathcal{L}_S^{(4)} = \frac{1}{2(1+\tilde{\alpha}^2)} h_{(\mu\nu)} \left[E^{\epsilon\mu}E^{\nu\delta} + \frac{4(m_+ - m_-)}{m_+ m_-} E^{\mu\delta}\Box\theta^{\nu\delta} + \frac{\left(2\theta^{\mu\epsilon}\theta^{\nu\delta} - \theta^{\mu\nu}\theta^{\epsilon\delta}\right)\Box^2}{2m_+ m_-}\right] h_{(\epsilon\delta)}. \tag{57}$$

Thus, as in the last section, the requirement of invariance under local shifts proportional to each other effectively solders the fields $A_{\mu\nu}$ and $B_{\mu\nu}$ into one combination $h_{\mu\nu} = \tilde{\alpha} A_{\mu\nu} - B_{\mu\nu}$. By using $\tilde{\alpha} = \pm\sqrt{m_-/m_+}$, it is easy to check that each of the terms of (57) is invariant under the discrete symmetry $(m_+, m_-, \gamma_+, \gamma_-) \to (-m_-, -m_+, \gamma_-, \gamma_+)$ which interchanges $W_{\pm 2}^{(3)} \rightleftharpoons W_{\mp 2}^{(3)}$. Furthermore, the action (57) is invariant under the local symmetries $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \epsilon_{\mu\nu\lambda}\Lambda^\lambda$ inherited from $W_\pm^{(3)}$. The three terms in (57) correspond exactly to the quadratic truncation of the new massive gravity of [12] up to an overall constant:

$$2(1+\tilde{\alpha}^2)\mathcal{L}_S^{(4)} = \left[-\sqrt{-g}R + \frac{m_+ - m_-}{m_+ m_-}\epsilon^{\mu\nu\rho}\Gamma^\epsilon_{\mu\gamma}\partial_\nu\Gamma^\gamma_{\epsilon\rho} + \frac{\sqrt{-g}}{m_+ m_-}\left(R_{\mu\nu}R^{\nu\mu} - \frac{3}{8}R^2\right)\right]_{hh} . \tag{58}$$

In identifying (57) with (58) we have used $g_{\mu\nu} = \eta_{\mu\nu} + 2h_{(\mu\nu)}$ (or $e_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$). The second term in (58) is the quadratic truncation of the topologically massive gravity of [4]: $\mathcal{L}_{TPM} = \epsilon^{\mu\nu\rho}\Gamma^\epsilon_{\mu\gamma}\left[\partial_\nu\Gamma^\gamma_{\epsilon\rho} + (2/3)\Gamma^\gamma_{\nu\delta}\Gamma^\delta_{\rho\epsilon}\right]$.

5 Conclusion

In both cases of spin 1 and 2 theories in $D = 2 + 1$ we have shown that the simple addition of two first-order self-dual models (parity singlets) of opposite helicities leads us to a parity invariant theory (for equal masses) which describes a parity doublet by means of a single field. Those are the well known Proca and Fierz-Pauli theories respectively. On the other hand, the addition of self-dual models with gauge symmetry demands extra interference terms between the opposite helicity states in order to produce the desired result. We have shown here that the generalized soldering furnishes those required terms in a systematic way also for spin 2 particles in a complete off-shell procedure. In particular, the Fierz-Pauli theory with its nontrivial mass term is automatically produced, section 3, out of two second-order self-dual models of opposite helicities which are the spin 2 analogues of the Maxwell-Chern-Simon theories.

In section 3 we have shown that if we start with two spin 2 self-dual models of third-order (quadratic truncation of topologically massive gravity) we end up, after soldering, exactly with the new massive gravity theory of [12]. Since in previous examples [6], the theories related via soldering turn out to be equivalent (up to contact terms in the correlation functions), our results suggest that there might be a local dual map between the gauge invariant sectors of $W^{(3)}_{+2} + W^{(3)}_{-2}$ and the new massive gravity theory [12] at linearized level. In particular, both theories have the same $m \to \infty$ limit (pure Einstein-Hilbert) contrary to the Fierz-Pauli theory (see discussion in [19]).

Extensions of the soldering formalism beyond the linear level in $D = 2 + 1$ as well as the introduction of interactions in the soldering of the third-order self-dual models are currently under investigation both in the soldering and master action approaches. Moreover, it would be interesting to investigate, see also [20, 21], higher spin ($s \geq 3$) generalizations of the soldering procedure in $D = 2 + 1$ and their possible relationships with massless higher spin theories in $D = 3 + 1$.

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References

[1] M. Stone, Illinois preprint, ILL - (TH) - 89-23, 1989; Phys. Rev. Lett. **63** (1989) 731; Nucl. Phys. B **327** (1989) 399

[2] R. Amorim, A. Das and C. Wotzasek, Phys. Rev. D **53** (1996) 5810.

[3] E. M. C. Abreu, R. Banerjee and C. Wotzasek, Nucl. Phys. B **509** (1998) 519.

[4] S. Deser, R. Jackiw and S. Templeton, Ann. of Phys. **140**(1982) 372.

[5] R. Banerjee and S. Kumar, Phys. Rev. D **60**, (1999) 085005.

[6] D. Dalmazi, A. de Souza Dutra and E. M. C. Abreu, Phys.Rev.D **74**025015 (2006); [Erratum-ibid. Phys. Rev. D**79** 109902 (2009)].

[7] M. Fierz, Helv. Phys. Acta **12** (1939) 3; M. Fierz, W. Pauli, Proc. Roy. Soc. **173** (1939) 211.

[8] A. Ilha and C. Wotzasek, Phys. Rev. D **63**, (2001) 105013.

[9] P.K. Townsend, K. Pilch and P. van Nieuwenhuizen, Phys. Lett B **136** (1984)38.

[10] C. Aragone and A. Khoudeir, Phys. Lett. B**173** 141 (1986).

[11] S. Deser and J. McCarthy, Phys. Lett. B**246** 441 (1990).

[12] E. Bergshoeff, O. Hohm and P.K. Townsend, "Massive Gravity in Three Dimensions", arXiv:0905.1259 [hep-th].

[13] M. Nakazone and I. Oda, "On Unitarity of Massive Gravity in Three Dimensions. ", arXiv:0902.3531 [hep-th].

[14] S.Deser and R. Jackiw, Phys.Lett.B **139** (1984) 371.

[15] R. Gaitan, "On the Coupling Problem of Higher Spin Fields in 2+1 Dimension ", PhD thesis, in spanish, arXiv:0711.2498.

[16] D.Dalmazi and E.L.Mendonça, Phys.Rev.D**79** 045025 (2009).

[17] C. Aragone, P.J. Arias and A. Khoudeir, Nuovo Cim.B**109**, 303-310 (1994).

[18] T. Scaria, "Studies in certain planar field theories", PhD Thesis, hep-th/0407022.

[19] S. Deser, gr-qc/9211010.

[20] T. Damour, S. Deser, Annales Poincare Phys.Theor. **47** 277 (1987).

[21] C. Aragone , A. Khoudeir,"Massive triadic Chern-Simons spin 3 theory", Presented at 8th Latin American Symposium on Relativity and Gravitation (SILARG), Sao Paulo, Brazil, 25-30 Jul 1993. Published in SILARG 1993:0529-533, hep-th/9310048.